

PARABOLIC FLOW FOR GENERALIZED COMPLEX MONGE-AMPÈRE TYPE EQUATIONS

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ABSTRACT. We study the parabolic flow for generalized complex Monge-Ampère type equations on closed Hermitian manifolds. We derive *a priori* C^∞ estimates for normalized solutions, and then prove the C^∞ convergence.

1. INTRODUCTION

Let (M, ω) be a compact Hermitian manifold of complex dimension $n \geq 2$ and χ another smooth Hermitian metric. We are interested in looking for a Hermitian metric χ' in the class $[\chi]$ satisfying

$$(1.1) \quad \chi'^n = \psi \sum_{\alpha=1}^n b_\alpha \chi'^{n-\alpha} \wedge \omega^\alpha,$$

where ψ is a smooth positive real function on M , b_α 's are nonnegative real constants, and $\sum_{\alpha=1}^n b_\alpha > 0$. This type of equations contains some of the most important ones in complex geometry and analysis.

The most known case is probably the complex Monge-Ampère equation,

$$(1.2) \quad \chi'^n = \psi \omega^n.$$

On closed Kähler manifolds, this equation is equivalent to the Calabi conjecture [3, 4]. In the famous work of Yau [41] (see also [1]), he proved the conjecture by solving the complex Monge-Ampère equation. A corollary of the Calabi conjecture is the existence of Kähler-Einstein metrics when the first Chern class is zero. Later, the result was extended to closed Hermitian manifolds, which was done by Cherrier [9], Tosatti and Weinkove [34, 35]. In [35], Tosatti and Weinkove stated a Hermitian version of Calabi conjecture, which used the first Bott-Chern class instead.

All of these works are based on continuity method, while the parabolic flow method is also a powerful measure. Cao [5] used Kähler-Ricci flow to reproduce the result of

Yau [41], and Gill [16] introduced Chern-Ricci flow to reproduce that of Tosatti and Weinkove [34, 35].

Another interesting case is Donaldson's problem,

$$(1.3) \quad \chi'^m = \psi \chi'^{m-1} \wedge \omega.$$

It was first proposed by Donaldson [11] in the study of moment maps. Chen [7] found out the same equation when he study the Mabuchi energy on Fano manifolds. The solvability implies a lower bound of the Mabuchi energy, and χ' is the critical metric. When ψ is a constant, the equation was studied by Chen [7, 8], Weinkove [39, 40], Song and Weinkove [26] using the J -flow. These results were extended by Fang, Lai and Ma [13] also by a parabolic flow. In [14, 15], Fang and Lai studied some other parabolic flows, one of which was adopted by the author [29] to study more general cases.

For Donaldson's problem, continuity method seems harder to apply. The author [28] introduced piecewise continuity method to solve the complex Monge-Ampère type equations on Hermitian manifolds, which contains Donaldson's equation. Li, Shi and Yao [23] reproduced the result of Song and Weinkove [26] by constructing a class of Kähler metrics. Admitting the result of Yau [41], Collins and Székelyhidi [10] were also able to solve the equation by a different argument. In the argument, they construct a class of equations in form (1.1) to reach Donaldson's equation from the complex Monge-Ampère equation. The approach was applied in more general equations by Székelyhidi [33].

For general cases of equation (1.1), the solvability was conjectured by Fang, Lai and Ma [13]. They solved the equation when the coefficients satisfies some special conditions. To investigate the numerical cone condition for J -flow conjectured in [22], Collins and Székelyhidi [10] studied equation (1.1). Later, the author [32] generalized the result to more general cases by piecewise continuity method.

In local coordinate charts, we may write ω and χ as

$$(1.4) \quad \omega = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

and

$$(1.5) \quad \chi = \frac{\sqrt{-1}}{2} \sum_{i,j} \chi_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

For convenience, we denote

$$(1.6) \quad \chi_u = \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u = \frac{\sqrt{-1}}{2} \sum_{i,j} (\chi_{i\bar{j}} + u_{i\bar{j}}) dz^i \wedge d\bar{z}^j.$$

Therefore, equation (1.1) can be written in the form,

$$(1.7) \quad \chi_u^n = \psi \sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha, \quad \chi_u > 0.$$

This is a fully nonlinear elliptic equation. Since the cokernal is nontrivial, it is not solvable for all $\psi > 0$. Generally, we try to find a function u and a real number b such that

$$(1.8) \quad \begin{cases} \chi_u^n = e^b \psi \sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha, \\ \chi_u > 0, \quad \sup_M u = 0. \end{cases}$$

Only in a few cases, we know $b = 0$ in advance. If χ and ω are both Kähler and ψ is constant, ψ is uniquely determined by

$$(1.9) \quad \psi = c := \frac{\int_M \chi^n}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha}$$

To reproduce the results in [32], we consider a parabolic flow for generalized complex Monge-Ampère type equations,

$$(1.10) \quad \frac{\partial u}{\partial t} = \ln \frac{\chi_u^n}{\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha} - \ln \psi$$

with initial value $u(x, 0) = 0$, where $\psi \in C^\infty(M)$ is positive, and $\chi_u > 0$. Following [26, 13, 19], we assume that there is a C^2 function v satisfying

$$(1.11) \quad n\chi_v^{n-1} > \psi \sum_{\alpha=1}^{n-1} b_\alpha (n-\alpha) \chi_v^{n-\alpha-1} \wedge \omega^\alpha, \quad \text{and } \chi_v > 0,$$

which is called the cone condition.

In the study, the key step is probably the sharp C^2 estimate.

Theorem 1.1. *Let (M, ω) be a closed Hermitian manifold of complex dimension n and χ also a Hermitian metric. Suppose that the cone condition (1.11) holds true. Then there exists a long time solution u to equation (1.10). Moreover, there are uniform constants C and A such that*

$$(1.12) \quad \Delta u + \text{tr} \chi \leq C e^{A(u - \inf_{M \times [0, t]} u)}.$$

Remark 1.2. As shown by Collins and Székelyhidi [10], the cone condition can be defined in the viscosity sense.

The sharp C^2 estimate can help us to obtain C^0 estimate, which is probably the most difficult one in the argument. It is worth a mention that there is a direct uniform estimate for the elliptic equation by the author [30, 31, 32] and Székelyhidi [33] while that for parabolic flows is not available so far. Since there are troublesome torsion terms, we need to apply a trick due to Phong and Sturm [25] with making a particular perfect square (see also [36]). When C^0 and C^2 estimates are available, all others can be obtained by Evans-Krylov theorem [12, 21, 37, 38] and Schauder theory.

To reproduce the results in [32], we need to discover some convergence property. However, it is very likely that $u(x, t)$ itself is divergent. Instead, we study the convergence of the normalized solution,

$$(1.13) \quad \tilde{u} = u - \frac{\int_M u \omega^n}{\int_M \omega^n}.$$

For general Hermitian manifolds, we have the following result.

Theorem 1.3. *Let (M, ω) be a closed Hermitian manifold of complex dimension n and χ also a Hermitian metric. Suppose that the cone condition (1.11) holds true. Then there exists a uniform constant C such that for all time $t \geq 0$,*

$$(1.14) \quad \sup_{x \in M} u(x, t) - \inf_{x \in M} u(x, t) < C,$$

given that

$$(1.15) \quad \chi^n \leq \psi \sum_{\alpha=1}^n b_\alpha \chi^{n-\alpha} \wedge \omega^\alpha.$$

As a consequence, \tilde{u} is C^∞ convergent to a smooth function \tilde{u}_∞ . Moreover, there is a unique real number b such that the pair $(\tilde{u}_\infty - \sup_M \tilde{u}_\infty, b)$ solves equation (1.8).

A corollary follows from Theorem 1.3.

Corollary 1.4. *Let (M, ω) be a closed Hermitian manifold of complex dimension n and χ also a Hermitian metric. Suppose that the cone condition (1.11) holds true, and $b_1 \cdots, b_{n-1}$ and ψ are fixed. Then there is a constant $K \geq 0$ such that if $c_n \geq K$ there exists a uniform constant C such that for all time $t \geq 0$,*

$$(1.16) \quad \sup_{x \in M} u(x, t) - \inf_{x \in M} u(x, t) < C,$$

As a consequence, \tilde{u} is C^∞ convergent to a smooth function \tilde{u}_∞ . Moreover, there is a unique real number b such that the pair $(\tilde{u}_\infty - \sup_M \tilde{u}_\infty, b)$ solves equation (1.8).

Should we have more knowledge of the manifold, it would be possible to obtain stronger results. When χ and ω are both Kähler, we have the following result.

Theorem 1.5. *Let (M, ω) be a closed Kähler manifold of complex dimension n and χ also a Kähler metric. Suppose that the cone condition (1.11) holds true and $\psi \geq c$ for all $x \in M$, where c is defined in (1.9). Then there exists a uniform constant C such that for all time $t \geq 0$,*

$$(1.17) \quad \sup_{x \in M} u(x, t) - \inf_{x \in M} u(x, t) < C.$$

Consequently, \tilde{u} is C^∞ convergent to a smooth function \tilde{u}_∞ . Moreover, there is a unique real number b such that the pair $(\tilde{u}_\infty - \sup_M \tilde{u}_\infty, b)$ solves equation (1.8).

It is worth a mention that we only require the concavity of the elliptic part in several key steps. So our argument can be applied to some other parabolic flows for generalized complex Monge-Ampère type equations. Fang and Lai [15] discussed different parabolic flows for complex Monge-Ampère type equations. Collins and Székelyhidi [10] used the equation

$$(1.18) \quad \frac{\partial u}{\partial t} = - \frac{\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha}{\chi_u^n}.$$

For simplicity, we shall consider an equivalent form analogous to J -flow when χ and ω are Kähler. We may call it generalized J -flow.

$$(1.19) \quad \frac{\partial u}{\partial t} = \frac{1}{c} - \frac{\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha}{\chi_u^n}.$$

Theorem 1.6. *Let (M, ω) be a closed Kähler manifold of complex dimension n and χ also a Kähler metric. Suppose that the cone condition (1.11) holds true for $\psi = c$. Then there exists a long time solution u to equation (1.19). Moreover, u is C^∞ convergent to a smooth function u_∞ . Moreover, $u_\infty - \sup_M u_\infty$ solves*

$$(1.20) \quad \begin{cases} \chi_u^n = c \sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha, \\ \chi_u > 0, \quad \sup_M u = 0. \end{cases}$$

The paper is organized as follows. In Section 2, we state some preliminary knowledge. In Section 3, we generalize the C^2 estimate of the author [29] and prove Theorem 1.1. In Section 4, time-independent estimates are derived from the C^2 estimate in Section 3. We prove the time-independent estimates for Hermitian manifolds and Kähler manifolds respectively. In Section 5, we give a new direct argument for gradient estimates. In Section 6, we briefly review the convergence of the normalized solution, and finish the proofs of Theorem 1.3 and Theorem 1.5. In Section 7, we study another parabolic flow used by Collins and Székelyhidi [10] and prove Theorem 1.6. We also discuss the lower bound and properness of the more general J -functional in the previous sections. It is pointed out by Gabor Székelyhidi that this is used in their paper [10].

2. PRELIMINARY

2.1. Notations. We denote by ∇ the Chern connection of g . As in [19], we express

$$(2.1) \quad X := \chi_u,$$

and thus in local coordinates

$$(2.2) \quad X_{i\bar{j}} = \chi_{i\bar{j}} + \bar{\partial}_{\bar{j}} \partial_i u.$$

Also, we denote the coefficients of X^{-1} by $X^{i\bar{j}}$.

Let $S_\alpha(\lambda)$ denote the α -th elementary symmetric polynomial of $\lambda \in \mathbb{R}^n$,

$$(2.3) \quad S_\alpha(\boldsymbol{\lambda}) = \sum_{1 \leq i_1 < \dots < i_\alpha \leq n} \lambda_{i_1} \cdots \lambda_{i_\alpha}.$$

For a nonsingular square matrix A , we define $S_\alpha(A) = S_\alpha(\boldsymbol{\lambda}(A))$ where $\boldsymbol{\lambda}(A)$ denote the eigenvalues of A . Further, write $S_\alpha(X) = S_\alpha(\boldsymbol{\lambda}_*(X))$ and $S_\alpha(X^{-1}) = S_\alpha(\boldsymbol{\lambda}^*(X^{-1}))$ where $\boldsymbol{\lambda}_*(A)$ and $\boldsymbol{\lambda}^*(A)$ denote the eigenvalues of a Hermitian matrix A with respect to $\{g_{i\bar{j}}\}$ and to $\{g^{i\bar{j}}\}$, respectively. In this paper, we shall use S_α to denote $S_\alpha(X^{-1})$. In local coordinates, equation (1.10) can be written in the form

$$(2.4) \quad \frac{\partial u}{\partial t} = \ln \frac{S_n(X)}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(X)} - \ln \psi,$$

where

$$(2.5) \quad c_\alpha = \frac{b_\alpha(n-\alpha)!\alpha!}{n!} = \frac{b_\alpha}{C_n^\alpha}.$$

2.2. Concavity. Concavity (or convexity) is an important assumption in the theory of fully nonlinear elliptic and parabolic equations, e.g. Evans-Krylov theorem. Indeed, the concavity itself will play a key role in the convergence later.

By the work of Caffarelli, Nirenberg and Spruck [2], we only need to prove that $\ln \frac{S_n(\lambda)}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda)}$ is concave with respect to λ instead of Hermitian matrix. For convenience, rewrite

$$(2.6) \quad \ln \frac{S_n(\lambda)}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda)} = \ln S_n(\lambda) - \ln \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda).$$

Define

$$(2.7) \quad B_{ij} = \frac{\partial^2}{\partial \lambda_j \partial \lambda_i} \left(\ln S_n(\lambda) - \ln \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda) \right).$$

We need to show that the matrix $\{B_{ij}\}_{n \times n}$ is non-positive definite.

Differentiating $\ln S_n(\lambda) - \ln \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda)$ twice,

$$(2.8) \quad \frac{\partial}{\partial \lambda_i} \left(\ln S_n(\lambda) - \ln \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda) \right) = \frac{S_{n-1;i}}{S_n} - \frac{\sum_{\alpha=1}^{n-1} c_\alpha S_{n-\alpha-1;i}(\lambda)}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda)},$$

and

$$(2.9) \quad \begin{aligned} & \frac{\partial^2}{\partial \lambda_j \partial \lambda_i} \left(\ln S_n(\lambda) - \ln \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda) \right) \\ &= \frac{S_{n-2;ij}(\lambda)}{S_n(\lambda)} - \frac{S_{n-1;i}(\lambda) S_{n-1;j}(\lambda)}{S_n^2(\lambda)} - \frac{\sum_{\alpha=1}^{n-2} c_\alpha S_{n-\alpha-2;ij}(\lambda)}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda)} \\ & \quad + \frac{\sum_{\alpha=1}^{n-1} c_\alpha S_{n-\alpha-1;i}(\lambda) \sum_{\alpha=1}^{n-1} c_\alpha S_{n-\alpha-1;j}(\lambda)}{\left(\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\lambda) \right)^2}. \end{aligned}$$

It is well known that $\ln S_n(\lambda) - \ln S_{n-\alpha}(\lambda)$ is concave, and consequently

$$(2.10) \quad \begin{aligned} & \left\{ \frac{S_{n-2;ij}(\lambda)}{S_n(\lambda)} - \frac{S_{n-1;i}(\lambda) S_{n-1;j}(\lambda)}{S_n^2(\lambda)} \right\}_{n \times n} \\ & \leq \left\{ \frac{S_{n-\alpha-2;ij}(\lambda)}{S_{n-\alpha}(\lambda)} - \frac{S_{n-\alpha-1;i}(\lambda) S_{n-\alpha-1;j}(\lambda)}{S_{n-\alpha}^2(\lambda)} \right\}_{n \times n}. \end{aligned}$$

When $\alpha = n$

$$(2.11) \quad \left\{ \frac{S_{n-2;ij}(\lambda)}{S_n(\lambda)} - \frac{S_{n-1;i}(\lambda) S_{n-1;j}(\lambda)}{S_n^2(\lambda)} \right\}_{n \times n} \leq \mathbf{0},$$

and when $\alpha = n - 1$

$$(2.12) \quad \left\{ \frac{S_{n-2;ij}(\boldsymbol{\lambda})}{S_n(\boldsymbol{\lambda})} - \frac{S_{n-1;i}(\boldsymbol{\lambda})S_{n-1;j}(\boldsymbol{\lambda})}{S_n^2(\boldsymbol{\lambda})} \right\}_{n \times n} \leq \left\{ -\frac{1}{S_1^2(\boldsymbol{\lambda})} \right\}_{n \times n}.$$

Hence,

$$(2.13) \quad \begin{aligned} \{B_{ij}\}_{n \times n} &\leq \left\{ \sum_{\alpha=1}^n \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \left(\frac{S_{n-\alpha-2;ij}}{S_{n-\alpha}} - \frac{S_{n-\alpha-1;i}S_{n-\alpha-1;j}}{S_{n-\alpha}^2} \right) \right. \\ &\quad \left. - \frac{\sum_{\alpha=1}^{n-2} c_\alpha S_{n-\alpha-2;ij}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} + \frac{\sum_{\alpha=1}^{n-1} c_\alpha S_{n-\alpha-1;i} \sum_{\alpha=1}^{n-1} c_\alpha S_{n-\alpha-1;j}}{\left(\sum_{\alpha=1}^n c_\alpha S_{n-\alpha} \right)^2} \right\}_{n \times n} \\ &= \left\{ - \sum_{\alpha=1}^n \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \frac{S_{n-\alpha-1;i}}{S_{n-\alpha}} \frac{S_{n-\alpha-1;j}}{S_{n-\alpha}} \right. \\ &\quad \left. + \sum_{\alpha=1}^{n-1} \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \frac{S_{n-\alpha-1;i}}{S_{n-\alpha}} \sum_{\alpha=1}^{n-1} \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \frac{S_{n-\alpha-1;j}}{S_{n-\alpha}} \right\}_{n \times n} \end{aligned}$$

For any real vector $\boldsymbol{\eta} = (\eta^1, \dots, \eta^n)$,

$$(2.14) \quad \begin{aligned} \sum_{i,j} B_{ij} \eta^i \eta^j &\leq - \sum_{\alpha=1}^n \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \left| \sum_i \frac{S_{n-\alpha-1;i} \eta^i}{S_{n-\alpha}} \right|^2 \\ &\quad + \left| \sum_{\alpha=1}^n \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \sum_i \frac{S_{n-\alpha-1;i} \eta^i}{S_{n-\alpha}} \right|^2, \end{aligned}$$

where $S_{-1;i} = 0$ by convention. Applying Hölder inequality,

$$(2.15) \quad \left| \sum_{\alpha=1}^n \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \sum_i \frac{S_{n-\alpha-1;i} \eta^i}{S_{n-\alpha}} \right|^2 \leq \sum_{\alpha=1}^n \frac{c_\alpha S_{n-\alpha}}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}} \left| \sum_i \frac{S_{n-\alpha-1;i} \eta^i}{S_{n-\alpha}} \right|^2,$$

and thus $\sum_{i,j} B_{ij} \eta^i \eta^j \leq 0$.

2.3. Formulas. We state some useful formulas at a given point p . Assume that at the point p , $g_{i\bar{j}} = \delta_{ij}$ and $X_{i\bar{j}}$ is diagonal in a specific chart. In this paper, we call such local coordinates normal coordinate charts around p .

We recall the formula from [18],

$$(2.16) \quad \overline{X_{i\bar{j}k}} = X_{j\bar{i}k},$$

and

$$(2.17) \quad \begin{aligned} X_{\bar{i}\bar{j}\bar{j}} - X_{j\bar{j}\bar{i}} &= R_{j\bar{j}\bar{i}}X_{\bar{i}} - R_{\bar{i}\bar{j}\bar{j}}X_{j\bar{j}} + 2\Re\left\{\sum_p \overline{T_{ij}^p}X_{i\bar{p}j}\right\} \\ &\quad - \sum_p T_{ij}^p \overline{T_{ij}^p}X_{p\bar{p}} - G_{\bar{i}\bar{j}\bar{j}}, \end{aligned}$$

where

$$(2.18) \quad \begin{aligned} G_{\bar{i}\bar{j}\bar{j}} &= \chi_{j\bar{j}\bar{i}} - \chi_{\bar{i}\bar{j}\bar{j}} + \sum_p R_{j\bar{j}i\bar{p}}\chi_{p\bar{i}} - \sum_p R_{\bar{i}\bar{j}p\bar{j}}\chi_{p\bar{j}} \\ &\quad + 2\Re\left\{\sum_p \overline{T_{ij}^p}\chi_{i\bar{p}j}\right\} - \sum_{p,q} T_{ij}^p \overline{T_{ij}^q}\chi_{p\bar{q}}. \end{aligned}$$

For convenience, we can rewrite equation (1.10) in the form

$$(2.19) \quad \partial_t u = -\ln \sum_{\alpha=1}^n c_\alpha S_\alpha - \ln \psi,$$

and condition (1.11) in the form

$$(2.20) \quad \sum_{\alpha=1}^{n-1} c_\alpha S_\alpha ((\chi_v|k)^{-1}) < \frac{1}{\psi},$$

for all k , where $(\chi_v|k)$ denotes the (k, k) minor matrix of χ_v .

Differentiating the equation at p

$$(2.21) \quad \partial_t(\partial_t u) = \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{\bar{i}\bar{i}})^2 (\partial_t u)_{\bar{i}\bar{i}},$$

$$(2.22) \quad \partial_t u_l = \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{\bar{i}\bar{i}})^2 X_{\bar{i}\bar{l}} - \frac{\partial_t \psi}{\psi},$$

and

$$\begin{aligned}
(2.23) \quad \partial_t u_{l\bar{l}} = & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[\sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{i}l\bar{l}} - \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{j}l} X_{i\bar{j}l} \right. \\
& - \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{j}l} X_{i\bar{j}l} \\
& + \sum_{i \neq j} S_{\alpha-2;ij} (X^{i\bar{i}})^2 (X^{j\bar{j}})^2 X_{j\bar{j}l} X_{i\bar{j}l} \\
& \left. - \sum_{i \neq j} S_{\alpha-2;ij} (X^{i\bar{i}})^2 (X^{j\bar{j}})^2 X_{j\bar{j}l} X_{i\bar{i}l} \right] \\
& + \left| \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{i}l\bar{l}} \right|^2 - \frac{\bar{\partial}_l \partial_l \psi}{\psi} + \frac{\partial_l \psi \bar{\partial}_l \psi}{\psi}.
\end{aligned}$$

Recall that the fundamental polynomials have strong concavity in Γ_n , which was shown in [20, 13].

Theorem 2.1. *For $\lambda \in \Gamma_n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, We have*

$$(2.24) \quad \sum_{i=1}^n \frac{S_{\alpha-1;i}(\lambda)}{\lambda_i} \xi_i \bar{\xi}_i + \sum_{i,j} S_{\alpha-2;ij}(\lambda) \xi_i \bar{\xi}_j \geq \sum_{i,j} \frac{S_{\alpha-1;i}(\lambda) S_{\alpha-1;j}(\lambda)}{S_\alpha(\lambda)} \xi_i \bar{\xi}_j \geq 0.$$

So we can control some terms in the right of (2.23),

$$\begin{aligned}
(2.25) \quad \partial_t X_{l\bar{l}} \leq & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[\sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{i}l\bar{l}} - \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{j}l} X_{i\bar{j}l} \right] \\
& - \sum_{\alpha=1}^n \frac{c_\alpha S_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left| \partial_l (\ln S_\alpha) \right|^2 + \left| \sum_{\alpha=1}^n \frac{c_\alpha S_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \partial_l (\ln S_\alpha) \right|^2 \\
& - \frac{\bar{\partial}_l \partial_l \psi}{\psi} + \frac{\partial_l \psi \bar{\partial}_l \psi}{\psi}.
\end{aligned}$$

Applying Hölder inequality to (2.25), and then summing it over l ,

$$\begin{aligned}
(2.26) \quad \partial_t w \leq & C_1 + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[\sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{i}l\bar{l}} \right. \\
& \left. - \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{j}l} X_{i\bar{j}l} \right].
\end{aligned}$$

Since $v \in C^2(M)$ and $\chi_v > 0$, we may assume

$$(2.27) \quad \epsilon\omega \leq \chi_v \leq \epsilon^{-1}\omega$$

for some $\epsilon > 0$.

Moreover, by the maximum principle, $\partial_t u$ attains its extremal values at $t = 0$,

$$(2.28) \quad \inf_{M \times \{0\}} \partial_t u \leq \partial_t u \leq \sup_{M \times \{0\}} \partial_t u.$$

It follows immediately that

$$(2.29) \quad |\partial_t \tilde{u}| \leq \sup_{M \times \{0\}} \partial_t u - \inf_{M \times \{0\}} \partial_t u,$$

and

$$(2.30) \quad \inf_M \frac{S_n(\chi)}{\psi \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\chi)} \leq \frac{S_n(X)}{\psi \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(X)} \leq \sup_M \frac{S_n(\chi)}{\psi \sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\chi)}.$$

Therefore, the flow remains Hermitian at any time.

3. THE SECOND ORDER ESTIMATE

In this section we derive the partial second order estimate for admissible solutions.

In order to prove the second order estimate, we need a key inequality. Guan [17], Collins and Székelyhidi [10] and Székelyhidi [33] have more general statements for elliptic equations.

Lemma 3.1. *There are constants $N, \theta > 0$ such that when $w \geq N$ at a point p ,*

$$(3.1) \quad \begin{aligned} & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\ & \leq -\ln \left(\psi \sum_{\alpha=1}^n c_\alpha S_\alpha \right) - \theta - \theta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2. \end{aligned}$$

Proof. Without loss of generality, we may assume that $X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}}$. Direct calculation shows that

$$\begin{aligned}
 & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 (3.2) \quad & \leq \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} X^{i\bar{i}} - \epsilon \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
 & = \left(1 - \frac{\epsilon}{2n} S_1\right) \sum_{\alpha=1}^n \frac{c_\alpha \alpha S_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} - \frac{\epsilon}{2} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

Noting that for all $s > 0$

$$(3.3) \quad \sup_M \psi s - \ln s \geq 1 + \ln \psi,$$

and thus if $S_1 \geq \frac{2n}{\epsilon}(1 + \sup_M \psi)$, we have

$$\begin{aligned}
 & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 (3.4) \quad & \leq -\ln \sum_{\alpha=1}^n c_\alpha S_\alpha - \ln \psi - 1 - \frac{\epsilon}{2} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

Now we just need to consider the case $X_{1\bar{1}} \geq \cdots \geq X_{n\bar{n}} \geq \frac{\epsilon}{2n(1+\sup_M \psi)}$. Rewriting

$$\begin{aligned}
 & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 (3.5) \quad & \leq \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_{i=2}^n S_{\alpha-1;i} (X^{i\bar{i}})^2 (X_{i\bar{i}} - (\chi_{i\bar{i}} + v_{i\bar{i}} - \delta)) \\
 & \quad - \delta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} S_{\alpha-1;1} X^{1\bar{1}}.
 \end{aligned}$$

If $\delta > 0$ is small enough, $\chi - \delta g$ is still positive definite and satisfies (2.20). So there are $N' > 0$ and $\sigma > 0$ such that for all $\lambda > N'$

$$(3.6) \quad \ln \sum_{\alpha=1}^n c_\alpha S_\alpha (\chi'^{-1}) < -\ln \psi - \sigma,$$

where

$$(3.7) \quad \chi' = \left\{ \begin{array}{c} \lambda \\ (\chi_v - \delta g|1) \end{array} \right\}_{n \times n}.$$

Since $-\ln \sum_{\alpha=1}^n c_\alpha S_\alpha$ is concave,

$$\begin{aligned}
 & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 (3.8) \quad & \leq -\ln \sum_{\alpha=1}^n c_\alpha S_\alpha + \ln \sum_{\alpha=1}^n c_\alpha S_\alpha (\chi''^{-1}) - \delta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
 & \quad + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} S_{\alpha-1;1} X^{1\bar{1}},
 \end{aligned}$$

where

$$(3.9) \quad \chi'' = \left\{ \begin{array}{c} X_{1\bar{1}} \\ (\chi_v - \delta g|1) \end{array} \right\}_{n \times n}.$$

If $X_{1\bar{1}} > N'$,

$$\begin{aligned}
 & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 & < -\ln \sum_{\alpha=1}^n c_\alpha S_\alpha - \ln \psi - \sigma - \delta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
 (3.10) \quad & \quad + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} S_{\alpha-1;1} X^{1\bar{1}} \\
 & \leq -\ln \sum_{\alpha=1}^n c_\alpha S_\alpha - \ln \psi - \sigma - \delta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
 & \quad + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} C_{n-1}^{\alpha-1} \left(\frac{2n}{\epsilon} (1 + \sup_M \psi) \right)^{\alpha-1} X^{1\bar{1}}.
 \end{aligned}$$

Using the bound (2.30) and let $X_{1\bar{1}}$ be sufficiently large, we have

$$\begin{aligned}
 & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 (3.11) \quad & \leq -\ln \sum_{\alpha=1}^n c_\alpha S_\alpha - \ln \psi - \frac{\sigma}{2} - \delta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

□

Theorem 3.2. *Let $u \in C^4(M \times [0, T])$ be an admissible solution to equation (1.10) and $w = \Delta u + \text{tr} \chi$. Then there are uniform constants C and A such that*

$$(3.12) \quad w \leq C e^{A(u - \inf_{M \times [0, t]} u)},$$

where C, A depend only on geometric data.

Proof. We consider the function $w e^\phi$ where ϕ is to specified later. Suppose that $\ln w e^\phi$ achieves its maximum at some point $(p, t_0) \in M_t = M \times (0, t]$. Choose a local chart around p such that $g_{i\bar{j}} = \delta_{ij}$ and $X_{i\bar{j}}$ is diagonal at p when $t = t_0$. Therefore, we have at the point (p, t_0) ,

$$(3.13) \quad \frac{\partial_l w}{w} + \partial_l \phi = 0,$$

$$(3.14) \quad \frac{\bar{\partial}_l w}{w} + \bar{\partial}_l \phi = 0,$$

$$(3.15) \quad \frac{\partial_t w}{w} + \partial_t \phi \geq 0,$$

and

$$(3.16) \quad \frac{\bar{\partial}_l \partial_l w}{w} - \frac{|\partial_l w|^2}{w^2} + \bar{\partial}_l \partial_l \phi \leq 0.$$

Without loss of generality, we may assume that $w \gg 1$. Otherwise, the proof is finished.

Since

$$(3.17) \quad \begin{aligned} & \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} \left| X_{i\bar{j}l} - X_{j\bar{l}} \frac{\partial_l w}{w} - T_{i\bar{l}}^j X_{j\bar{j}} \right|^2 \\ &= \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{i\bar{j}l} X_{j\bar{l}} - \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \frac{|\partial_i w|^2}{w} \\ & \quad + \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{j\bar{j}} T_{i\bar{l}}^j \overline{T_{i\bar{l}}^j} + \frac{2}{w} \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i w \right\} \\ & \quad - 2 \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ X_{i\bar{j}l} \overline{T_{i\bar{l}}^j} \right\}, \end{aligned}$$

where \hat{T} denotes the torsion with respect to the Hermitian metric χ . Then

$$\begin{aligned}
 & -\frac{2}{w} \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i w \right\} + \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \frac{|\partial_i w|^2}{w} \\
 (3.18) \quad & \leq \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{i\bar{j}l} X_{j\bar{l}} - 2 \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ X_{i\bar{j}l} \bar{T}_{il}^j \right\} \\
 & \quad + \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{j\bar{j}} T_{il}^j \bar{T}_{il}^j.
 \end{aligned}$$

So by (2.17) and (3.18),

$$\begin{aligned}
 & \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{l}} X_{i\bar{j}l} - \sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{l}l} \\
 (3.19) \quad & \geq -\frac{2}{w} \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i w \right\} + \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \frac{|\partial_i w|^2}{w} \\
 & \quad - \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i w + \sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 \left(-R_{l\bar{l}i\bar{i}} X_{i\bar{i}} + R_{i\bar{i}l\bar{l}} X_{l\bar{l}} + G_{i\bar{i}l\bar{l}} \right).
 \end{aligned}$$

Substituting (3.16) into (3.19)

$$\begin{aligned}
 & \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{l}} X_{i\bar{j}l} - \sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{l}l} \\
 (3.20) \quad & \geq -\frac{2}{w} \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i w \right\} + w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \\
 & \quad + \sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 \left(-R_{l\bar{l}i\bar{i}} X_{i\bar{i}} + R_{i\bar{i}l\bar{l}} X_{l\bar{l}} + G_{i\bar{i}l\bar{l}} \right).
 \end{aligned}$$

Combining (2.26) and (3.20),

$$\begin{aligned}
 \partial_t w & \leq C_1 + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[\frac{2}{w} \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i w \right\} \right. \\
 (3.21) \quad & \quad \left. - w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \right] \\
 & \quad - \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[\sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 \left(-R_{l\bar{l}i\bar{i}} X_{i\bar{i}} + R_{i\bar{i}l\bar{l}} X_{l\bar{l}} + G_{i\bar{i}l\bar{l}} \right) \right].
 \end{aligned}$$

Then

$$\begin{aligned}
(3.22) \quad \partial_t w \leq & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[\frac{2}{w} \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i w \right\} \right. \\
& \left. - w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \right] \\
& + C_3 w \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_2.
\end{aligned}$$

By (3.14) and (3.15),

$$\begin{aligned}
(3.23) \quad w \partial_t \phi \geq & \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \left[2 \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i \phi \right\} \right. \\
& \left. + w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \right] \\
& - C_3 w \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 - C_2.
\end{aligned}$$

To apply a trick due to Phong and Sturm [25], we specify ϕ as follows,

$$(3.24) \quad \phi := -A(u - v) + \frac{1}{u - v - \inf_{M_t}(u - v) + 1} = -A(u - v) + E_1.$$

We may assume $A > N \gg 1$, where N is the crucial constant in Lemma 3.1.

It is easy to see that

$$(3.25) \quad \partial_t \phi = -(A + E_1^2) \partial_t u,$$

$$(3.26) \quad \partial_i \phi = -(A + E_1^2)(u_i - v_i),$$

and

$$(3.27) \quad \bar{\partial}_i \partial_i \phi = -(A + E_1^2)(u_{i\bar{i}} - v_{i\bar{i}}) + 2|u_i - v_i|^2 E_1^3.$$

By (3.26) and (3.27),

$$\begin{aligned}
(3.28) \quad & 2 \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i \phi \right\} + w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \\
& = -2(A + E_1^2) \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_{j,k} \hat{T}_{ij}^k \chi_{k\bar{j}} (u_{\bar{i}} - v_{\bar{i}}) \right\} \\
& \quad + 2E_1^3 w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |u_i - v_i|^2 - (A + E_1^2) w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}).
\end{aligned}$$

Using Schwarz inequality to control the first term in the right,

$$\begin{aligned}
(3.29) \quad & 2(A + E_1^2) \left| \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_{j,k} \hat{T}_{ij}^k \chi_{k\bar{j}} (u_{\bar{i}} - v_{\bar{i}}) \right\} \right| \\
& \leq w E_1^3 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |u_i - v_i|^2 + \frac{C_4 A^2}{w E_1^3} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2,
\end{aligned}$$

and hence

$$\begin{aligned}
(3.30) \quad & 2 \sum_{i,j} S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k \hat{T}_{ij}^k \chi_{k\bar{j}} \bar{\partial}_i \phi \right\} + w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \\
& \geq -\frac{C_4 A^2}{w E_1^3} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 - (A + E_1^2) w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}).
\end{aligned}$$

Substituting (3.25) and (3.30) into (3.23)

$$\begin{aligned}
(3.31) \quad & w(A + E_1^2) \partial_t u \leq \frac{C_4 A^2}{w E_1^3} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
& \quad + (A + E_1^2) w \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
& \quad + C_3 w \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_2,
\end{aligned}$$

that is

$$\begin{aligned}
 (3.32) \quad 0 \leq w(A + E_1^2) & \left[\ln \left(\psi \sum_{\alpha=1}^n c_\alpha S_\alpha \right) \right. \\
 & \left. + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \right] \\
 & + \left(\frac{C_4 A^2}{w E_1^3} + C_3 w \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_2.
 \end{aligned}$$

If $w > A(u - v - \inf_{M_t}(u - v) + 1)^{\frac{3}{2}} = A E_1^{-\frac{3}{2}}$, there is $\theta > 0$ such that,

$$\begin{aligned}
 (3.33) \quad 0 \leq -\theta w(A + E_1^2) & \left[1 + \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \right] \\
 & + w(C_3 + C_4) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_2.
 \end{aligned}$$

This gives a bound $w \leq 1$ at (p, t_0) if we pick a sufficiently large A , which contradicts our assumption.

On the other hand, if $w \leq A(u - v - \inf_{M_t}(u - v) + 1)^{\frac{3}{2}}$,

$$\begin{aligned}
 (3.34) \quad we^\phi & \leq we^\phi|_{(p, t_0)} \leq A(u - v - \inf_{M_t}(u - v) + 1)^{\frac{3}{2}} e^{-A(u-v)+1}|_{(p, t_0)} \\
 & \leq A e^2 e^{-A \inf_{M_t}(u-v)},
 \end{aligned}$$

and hence

$$\begin{aligned}
 (3.35) \quad w & \leq A e^2 e^{-A \inf_{M_t}(u-v) + A(u-v) - E_1} \\
 & \leq C e^{A(u - \inf_{M_t} u)}.
 \end{aligned}$$

□

4. LONG TIME EXISTENCE AND TIME-INDEPENDENT ESTIMATES

Since $\partial_t u$ is bounded, we are able to obtain C^0 estimate dependent on time t . By the Evans-Krylov theorem and Schauder estimates, we can obtain C^∞ estimates dependent on t . Then it is standard to prove the long time existence. To prove convergence, we need time-independent estimates instead. In this section, we obtain time-independent estimates up to second order. Higher order estimates follow from Evans-Krylov theorem and Schauder estimate.

4.1. Hermitian case.

Theorem 4.1. *Under the assumption of Theorem 1.3, there exists a uniform constant C such that*

$$(4.1) \quad \sup_M u(x, t) - \inf_M u(x, t) \leq C.$$

Proof. We prove the theorem by contradiction. If such a bound does not exist, there is a sequence $t_i \rightarrow \infty$ such that

$$(4.2) \quad \sup_M u(x, t_i) - \inf_M u(x, t_i) \rightarrow \infty.$$

By (1.15) and the maximum principle,

$$(4.3) \quad \partial_t u \leq 0.$$

Thus for $t > s \geq 0$,

$$(4.4) \quad \sup_M u(x, t) \leq \sup_M u(x, s) \leq 0$$

and

$$(4.5) \quad \inf_M u(x, t) \leq \inf_M u(x, s) \leq 0$$

So we have

$$(4.6) \quad \inf_M u(x, t_i) = \inf_{t \in [0, t_i]} \inf_M u(x, t) \rightarrow -\infty.$$

By theorem 3.2,

$$(4.7) \quad \begin{aligned} w(x, t_i) &\leq C e^{A(u(x, t_i) - \inf_{M \times [0, t_i]} u)} \\ &= C e^{A(u(x, t_i) - \inf_M u(x, t_i))}. \end{aligned}$$

As shown in [34], it follows that

$$(4.8) \quad \sup_M u(x, t_i) - \inf_M u(x, t_i) \leq C$$

for some positive constant C , which contradicts our assumption. □

Following Theorem 4.1, we can show that the C^2 estimate is also independent on time t .

Corollary 4.2. *Under the assumption of Theorem 1.3, there exists a uniform constant C such that*

$$(4.9) \quad w(x, t) \leq C.$$

Proof. By (4.5) and theorem 3.2,

$$(4.10) \quad w(x, t) \leq Ce^{A(u - \inf_{M \times [0, t]} u)} = Ce^{A(u - \inf_M u(x, t))}.$$

Then the conclusion follows from Theorem 4.1. □

4.2. Kähler case.

Theorem 4.3. *Under the assumption of Theorem 1.5, there exists a uniform constant C such that*

$$(4.11) \quad \sup_M u(x, t) - \inf_M u(x, t) \leq C.$$

First of all, we recall the definition of extended J -functionals, which was done in [13, 29]. Let

$$(4.12) \quad \mathcal{H} := \{u \in C^\infty(M) \mid \chi_u > 0\}.$$

For any curve $v(s) \in \mathcal{H}$, the functional J_α is defined by

$$(4.13) \quad \frac{dJ_\alpha}{ds} = \int_M \frac{\partial v}{\partial s} \chi_v^{n-\alpha} \wedge \omega^\alpha.$$

Then we have a formula for J_α of $u \in \mathcal{H}$,

$$(4.14) \quad J_\alpha(u) = \int_0^1 \int_M \frac{\partial v}{\partial s} \chi_v^{n-\alpha} \wedge \omega^\alpha ds,$$

for any path $v(s) \subset \mathcal{H}$ connecting 0 and u . The functional is independent on the path. Restricting the integration to the line $v(s) = su$,

$$(4.15) \quad \begin{aligned} J_\alpha(u) &= \int_0^1 \int_M u \chi_{su}^{n-\alpha} \wedge \omega^\alpha ds \\ &= \frac{1}{n - \alpha + 1} \sum_{i=0}^{n-\alpha} \int_M u \chi_u^i \wedge \chi^{n-\alpha-i} \wedge \omega^\alpha. \end{aligned}$$

Define a new functional J for equation (1.10),

$$(4.16) \quad J(u) := \sum_{\alpha=1}^n b_\alpha J_\alpha(u) = \int_0^1 \int_M \frac{\partial v}{\partial s} \left(\sum_{\alpha=1}^n b_\alpha \chi_v^{n-\alpha} \wedge \omega^\alpha \right) ds,$$

for any path $v(s) \subset \mathcal{H}$ connecting 0 and u . For any $u(x, t)$, we have

$$(4.17) \quad J(u(T)) = \int_0^T \int_M \frac{\partial u}{\partial t} \left(\sum_{\alpha=1}^n b_\alpha \chi_v^{n-\alpha} \wedge \omega^\alpha \right) dt = \int_0^T \frac{dJ}{dt} dt,$$

and thus along the solution flow $u(x, t)$ to equation (1.10),

$$(4.18) \quad \begin{aligned} \frac{d}{dt} J(u) &= \int_M \partial_t u \left(\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha \right) \\ &= \int_M \left(\ln \frac{\chi_u^n}{\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha} - \ln \psi \right) \left(\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha \right) \\ &\leq \ln c \int_M \left(\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha \right) - \int_M \ln \psi \left(\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha \right). \end{aligned}$$

The last inequality follows from Jensen's inequality. The condition $\psi \geq c$ implies that

$$(4.19) \quad \frac{d}{dt} J(u) \leq 0.$$

This tells us that $J(u)$ is decreasing and non-positive along the solution flow. Let

$$(4.20) \quad \hat{u} = u - \frac{J(u)}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha}.$$

Then

$$(4.21) \quad u(x, t) \leq \hat{u}(x, t).$$

Lemma 4.4.

$$(4.22) \quad 0 \leq \sup_M \hat{u}(x, t) \leq -C_1 \inf_M \hat{u}(x, t) + C_2,$$

for some constants C_1 and C_2 .

Proof. By (4.17),

$$(4.23) \quad J(\hat{u}(T)) = \int_0^T \int_M \frac{\partial \hat{u}}{\partial t} \left(\sum_{\alpha=1}^n b_\alpha \chi_u^{n-\alpha} \wedge \omega^\alpha \right) dt = 0.$$

Together with (4.15), we have

$$(4.24) \quad \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \sum_{i=0}^{n-\alpha} \int_M \hat{u} \chi_{\hat{u}}^i \wedge \chi^{n-\alpha-i} \wedge \omega^\alpha = 0.$$

The first inequality in (4.22) follows from (4.24).

Rewriting (4.24),

$$(4.25) \quad \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \int_M \hat{u} \chi^{n-\alpha} \wedge \omega^\alpha = - \sum_{\alpha=1}^{n-1} \frac{b_\alpha}{n-\alpha+1} \sum_{i=1}^{n-\alpha} \int_M \hat{u} \chi_{\hat{u}}^i \wedge \chi^{n-\alpha-i} \wedge \omega^\alpha.$$

Let C be a positive constant satisfying

$$(4.26) \quad \omega^n \leq C \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \chi^{n-\alpha} \wedge \omega^\alpha.$$

Then

$$(4.27) \quad \begin{aligned} \int_M \hat{u} \omega^n &= \int_M \left(\hat{u} - \inf_M \hat{u} \right) \omega^n + \int_M \inf_M \hat{u} \omega^n \\ &\leq C \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \int_M \left(\hat{u} - \inf_M \hat{u} \right) \chi^{n-\alpha} \wedge \omega^\alpha + \int_M \inf_M \hat{u} \omega^n \\ &= C \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \int_M \hat{u} \chi^{n-\alpha} \wedge \omega^\alpha \\ &\quad + \inf_M \hat{u} \left(\int_M \omega^n - C \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \int_M \chi^{n-\alpha} \wedge \omega^\alpha \right). \end{aligned}$$

Substituting (4.25) into (4.27),

$$(4.28) \quad \begin{aligned} \int_M \hat{u} \omega^n &\leq -C \sum_{\alpha=1}^{n-1} \frac{b_\alpha}{n-\alpha+1} \sum_{i=1}^{n-\alpha} \int_M \hat{u} \chi_{\hat{u}}^i \wedge \chi^{n-\alpha-i} \wedge \omega^\alpha \\ &\quad + \inf_M \hat{u} \left(\int_M \omega^n - C \sum_{\alpha=1}^n \frac{b_\alpha}{n-\alpha+1} \int_M \chi^{n-\alpha} \wedge \omega^\alpha \right) \\ &\leq \inf_M \hat{u} \left(\int_M \omega^n - C \sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha \right). \end{aligned}$$

As shown in [41], it implies the second inequality in (4.22). □

It remains to prove Theorem 4.3.

Proof of Theorem 4.3. By Lemma 4.4, it is sufficient to show a lower bound for $\inf_M \hat{u}(x, t)$. If such a lower bound does not exist, then we can choose a sequence $t_i \rightarrow \infty$ such that

$$(4.29) \quad \inf_M \hat{u}(x, t_i) = \inf_{t \in [0, t_i]} \inf_M \hat{u}(x, t),$$

and

$$(4.30) \quad \inf_M \hat{u}(x, t_i) \rightarrow -\infty.$$

By (4.29), for any $t \leq t_i$,

$$(4.31) \quad \begin{aligned} & \inf_M u(x, t_i) - \inf_M u(x, t) \\ &= \inf_M \hat{u}(x, t_i) - \inf_M \hat{u}(x, t) + \frac{J(u(x, t_i)) - J(u(x, t))}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha} \\ &\leq \frac{J(u(x, t_i)) - J(u(x, t))}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha}. \end{aligned}$$

Since $J(u)$ is decreasing along the solution flow, $\inf_M u(x, t_i) \leq \inf_M u(x, t)$. Together with (4.21), we have

$$(4.32) \quad \inf_M u(x, t_i) = \inf_{t \in [0, t_i]} \inf_M u(x, t),$$

and

$$(4.33) \quad \inf_M u(x, t_i) \rightarrow -\infty.$$

By theorem 3.2,

$$(4.34) \quad \begin{aligned} w(x, t_i) &\leq C e^{A(u(x, t_i) - \inf_{M \times [0, t_i]} u)} \\ &= C e^{A(u(x, t_i) - \inf_M u(x, t_i))}. \end{aligned}$$

As shown in [34], it follows that

$$(4.35) \quad \sup_M u(x, t_i) - \inf_M u(x, t_i) \leq C$$

for some positive constant C , which contradicts our assumption. \square

Following Theorem 4.3, we can show that the C^2 estimate is also independent on time t .

Corollary 4.5. *Under the assumption of Theorem 1.5, there exists a uniform constant C such that*

$$(4.36) \quad w(x, t) \leq C.$$

Proof. There exists $t_0 \in [0, t]$ such that $\inf_M u(x, t_0) = \inf_{M \times [0, t]} u(x, s)$, and hence

$$\begin{aligned}
 (4.37) \quad u(x, t) - \inf_{M \times [0, t]} u(x, s) &= u(x, t) - \inf_M u(x, t_0) \\
 &= \hat{u}(x, t) - \inf_M u(x, t_0) + \frac{J(u(x, t))}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha} \\
 &= \hat{u}(x, t) - \inf_M \hat{u}(x, t_0) + \frac{J(u(x, t)) - J(u(x, t_0))}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha} \\
 &\leq \hat{u}(x, t) - \inf_M \hat{u}(x, t_0).
 \end{aligned}$$

Together with theorem 3.2,

$$(4.38) \quad w(x, t) \leq C e^{A(u - \inf_{M \times [0, t]} u)} \leq C e^{A(\hat{u}(x, t) - \inf_M \hat{u}(x, t_0))}.$$

Then the conclusion follows from the fact that \hat{u} is uniformly bounded on $M \times [0, \infty)$, which is in the proof of Theorem 4.3. \square

5. THE GRADIENT ESTIMATE

Although C^0 estimate and partial C^2 estimate can lead to a gradient estimate, we provide a different argument following [19, 27].

Theorem 5.1. *Let $u \in C^4(M \times [0, T])$ be an admissible solution to equation (1.10) and $w = \Delta u + \text{tr} \chi$. Suppose that*

$$(5.1) \quad \mu(x, t) = u(x, t) + H(t), \quad \frac{dH}{dt} \geq 0.$$

Then there are uniform constants C and A such that

$$(5.2) \quad |\nabla u|^2 \leq C e^{A E_2}, \quad E_2 = e^{\sup_{M \times [0, t]} (\mu - v) - \inf_{M \times [0, t]} (\mu - v)} - e^{\sup_{M \times [0, t]} (\mu - v) - (\mu - v)},$$

where C, A depend only on geometric data.

Proof. Suppose that the function $|\nabla u|^2 e^\phi$ attains its maximum at $(p, t_0) \in M_t = M \times (0, t]$. As previous, we choose a normal coordinate chart around p . Without loss of generality, we may assume $|\nabla u| \gg 1$ and $|\nabla u| > |\nabla v|$ at (p, t_0) . Thus we have the following relations at (p, t_0) ,

$$(5.3) \quad \frac{\partial_i (|\nabla u|^2)}{|\nabla u|^2} + \partial_i \phi = 0,$$

$$(5.4) \quad \frac{\bar{\partial}_i(|\nabla u|^2)}{|\nabla u|^2} + \bar{\partial}_i \phi = 0,$$

$$(5.5) \quad \frac{\partial_t(|\nabla u|^2)}{|\nabla u|^2} + \partial_t \phi \geq 0,$$

and

$$(5.6) \quad \frac{\bar{\partial}_i \partial_i |\nabla u|^2}{|\nabla u|^2} - \frac{\partial_i(|\nabla u|^2) \bar{\partial}_i(|\nabla u|^2)}{|\nabla u|^4} + \bar{\partial}_i \partial_i \phi \leq 0.$$

Direct computation shows that,

$$(5.7) \quad \partial_i(|\nabla u|^2) = \sum_k (u_k u_{i\bar{k}} + u_{ki} u_{\bar{k}}),$$

$$(5.8) \quad \partial_t(|\nabla u|^2) = \sum_k (\partial_t u_k u_{\bar{k}} + u_k \partial_t u_{\bar{k}}),$$

and

$$(5.9) \quad \begin{aligned} \bar{\partial}_i \partial_i(|\nabla u|^2) &= \sum_k (u_{ki} u_{\bar{k}\bar{i}} + u_{i\bar{i}k} u_{\bar{k}} + u_{i\bar{i}\bar{k}} u_k) + \sum_{k,l} R_{i\bar{i}k\bar{l}} u_l u_{\bar{k}} \\ &\quad \sum_k \left| u_{\bar{k}i} - \sum_l T_{il}^k u_{\bar{l}} \right|^2 - \sum_k \left| \sum_l T_{il}^k u_{\bar{l}} \right|^2. \end{aligned}$$

By (5.3) and (5.8),

$$(5.10) \quad |\partial_i(|\nabla u|^2)|^2 = \left| \sum_k u_{ki} u_{\bar{k}} \right|^2 - 2|\nabla u|^2 \Re \left\{ \sum_k u_k u_{\bar{k}i} \bar{\partial}_i \phi \right\} - \left| \sum_k u_k u_{\bar{k}i} \right|^2.$$

By (5.9) and Schwarz inequality,

$$\begin{aligned}
& \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i (|\nabla u|^2) \\
& \geq \frac{1}{|\nabla u|^2} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \left| \sum_k u_{ki} u_{\bar{k}} \right|^2 - \sum_k \partial_k S_{\alpha} u_{\bar{k}} - \sum_k \bar{\partial}_k S_{\alpha} u_k \\
& \quad - \sum_{i,k} S_{\alpha-1;i} (X^{i\bar{i}})^2 \chi_{i\bar{i}k} u_{\bar{k}} - \sum_{i,k} S_{\alpha-1;i} (X^{i\bar{i}})^2 \chi_{i\bar{i}\bar{k}} u_k \\
& \quad + \sum_{i,k,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 R_{i\bar{i}k\bar{l}} u_l u_{\bar{k}} - \sum_{i,k} S_{\alpha-1;i} (X^{i\bar{i}})^2 \left| \sum_l T_{il}^k u_{\bar{l}} \right|^2 \\
& \geq \frac{1}{|\nabla u|^2} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \left| \sum_k u_{ki} u_{\bar{k}} \right|^2 - \sum_k \partial_k S_{\alpha} u_{\bar{k}} - \sum_k \bar{\partial}_k S_{\alpha} u_k \\
& \quad - C_2 |\nabla u|^2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 - C_2 |\nabla u| \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2,
\end{aligned} \tag{5.11}$$

for some uniform constant $C_2 > 0$; by (5.10),

$$\begin{aligned}
& \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\partial_i (|\nabla u|^2)|^2 \\
& \leq \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \left| \sum_k u_{ki} u_{\bar{k}} \right|^2 - 2 |\nabla u|^2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k u_k u_{\bar{k}i} \bar{\partial}_i \phi \right\}.
\end{aligned} \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.6),

$$\begin{aligned}
& |\nabla u|^2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi + 2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k u_k u_{\bar{k}i} \bar{\partial}_i \phi \right\} \\
& \leq \sum_k \partial_k S_{\alpha} u_{\bar{k}} + \sum_k \bar{\partial}_k S_{\alpha} u_k + C_2 (|\nabla u|^2 + |\nabla u|) \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
\end{aligned} \tag{5.13}$$

Let $\phi = Ae^{\eta}$, where $\eta = v - \mu + \sup_{M \times [0,t]} (\mu - v) \geq 0$ and A is to be determined. It is easy to see that

$$\partial_t \phi = \phi \partial_t \eta = -\phi \partial_t \mu, \tag{5.14}$$

$$\partial_i \phi = \phi \eta_i, \tag{5.15}$$

$$\bar{\partial}_i \phi = \phi \eta_{\bar{i}}, \tag{5.16}$$

and

$$\bar{\partial}_i \partial_i \phi = \phi (\eta_i \eta_{\bar{i}} + \eta_{i\bar{i}}). \tag{5.17}$$

Then from (5.13),

$$\begin{aligned}
 & |\nabla u|^2 \phi \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (\eta_i \eta_{\bar{i}} + \eta_{i\bar{i}}) + 2\phi \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k u_k u_{\bar{k}i} \eta_{\bar{i}} \right\} \\
 (5.18) \quad & \leq \sum_k \partial_k S_\alpha u_{\bar{k}} + \sum_k \bar{\partial}_k S_\alpha u_k + C_2 (|\nabla u|^2 + |\nabla u|) \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

Controlling the second term,

$$\begin{aligned}
 & 2\phi \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k u_k u_{\bar{k}i} \eta_{\bar{i}} \right\} \\
 & = 2\phi \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} - 2\phi \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \Re \left\{ \sum_k u_k \chi_{\bar{k}i} \eta_{\bar{i}} \right\} \\
 (5.19) \quad & \geq 2\phi \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} - \frac{\phi}{2} |\nabla u|^2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2 \\
 & \quad - C_3 \phi \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

Substituting (5.19) into (5.18),

$$\begin{aligned}
 0 \leq & \sum_k \partial_k S_\alpha u_{\bar{k}} + \sum_k \bar{\partial}_k S_\alpha u_k + \phi |\nabla u|^2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
 (5.20) \quad & - 2\phi \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} - \frac{\phi}{2} |\nabla u|^2 \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2 \\
 & + C_2 (|\nabla u|^2 + |\nabla u|) \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_3 \phi \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

Noting that

$$(5.21) \quad \partial_t (|\nabla u|^2) \leq C_4 |\nabla u| - \sum_k \partial_k \left(\ln \sum_{\alpha=1}^n c_\alpha S_\alpha \right) u_{\bar{k}} - \sum_k \bar{\partial}_k \left(\ln \sum_{\alpha=1}^n c_\alpha S_\alpha \right) u_k,$$

then by (5.5) and (5.14),

$$\begin{aligned}
 (5.22) \quad & \sum_k \partial_k \left(\ln \sum_{\alpha=1}^n c_\alpha S_\alpha \right) u_{\bar{k}} + \sum_k \bar{\partial}_k \left(\ln \sum_{\alpha=1}^n c_\alpha S_\alpha \right) u_k \leq C_4 |\nabla u| + |\nabla u|^2 \partial_t \phi \\
 & \leq C_4 |\nabla u| - \phi |\nabla u|^2 \partial_t u.
 \end{aligned}$$

Putting together (5.20) and (5.22),

$$\begin{aligned}
(5.23) \quad 0 \leq & \ln \left(\psi \sum_{\alpha=1}^n c_{\alpha} S_{\alpha} \right) + \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
& - \frac{2}{|\nabla u|^2} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} \\
& - \frac{1}{2} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2 \\
& + \frac{C_2(|\nabla u| + 1)}{\phi |\nabla u|} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
& + \frac{C_3}{|\nabla u|^2} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + \frac{C_4}{\phi |\nabla u|}.
\end{aligned}$$

By the fact that $\phi \geq A$,

$$\begin{aligned}
(5.24) \quad 0 \leq & \ln \left(\psi \sum_{\alpha=1}^n c_{\alpha} S_{\alpha} \right) + \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (u_{i\bar{i}} - v_{i\bar{i}}) \\
& - \frac{2}{|\nabla u|^2} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} \\
& - \frac{1}{2} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2 + \frac{C_4}{A |\nabla u|} \\
& + C_5 \left(\frac{1}{A} + \frac{1}{A |\nabla u|} + \frac{1}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
\end{aligned}$$

Since we assume $|\nabla u| > |\nabla v|$,

$$\begin{aligned}
(5.25) \quad & - \frac{2}{|\nabla u|^2} \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} \\
& \leq 4 \sum_{\alpha=1}^n \frac{c_{\alpha}}{\sum_{\alpha=1}^n c_{\alpha} S_{\alpha}} \sum_i S_{\alpha-1;i} X^{i\bar{i}} \leq 4n.
\end{aligned}$$

Case 1. $w > N$, where N is the crucial constant in Lemma 3.1. We have

$$\begin{aligned}
 & \theta + \theta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
 & + \frac{1}{2} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2 \\
 (5.26) \quad & \leq -\frac{2}{|\nabla u|^2} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re\{u_i \eta_{\bar{i}}\} + \frac{C_4}{A|\nabla u|} \\
 & + C_5 \left(\frac{1}{A} + \frac{1}{A|\nabla u|} + \frac{1}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

If $\sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} S_{\alpha-1;i} (X^{i\bar{i}})^2 \geq K$ for some i and $K > 0$,

$$\begin{aligned}
 & \theta + \frac{\theta}{2} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + \frac{\theta K}{2} \\
 (5.27) \quad & \leq 4n + \frac{C_4}{A|\nabla u|} + C_5 \left(\frac{1}{A} + \frac{1}{A|\nabla u|} + \frac{1}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

Choosing θ satisfying $\theta K > 8n$,

$$\begin{aligned}
 & \theta + \frac{\theta}{2} \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
 (5.28) \quad & \leq \frac{C_4}{A|\nabla u|} + C_5 \left(\frac{1}{A} + \frac{1}{A|\nabla u|} + \frac{1}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2.
 \end{aligned}$$

When A is large enough, we derive a bound $|\nabla u| \leq C$.

If $\sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} S_{\alpha-1;i} (X^{i\bar{i}})^2 \leq K$ for all i , then

$$(5.29) \quad \sum_{\alpha=1}^n \frac{c_\alpha \alpha S_\alpha S_1}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \leq n^2 K,$$

and hence

$$(5.30) \quad S_1 \leq n^2 K.$$

By Schwarz inequality and (5.30),

$$\begin{aligned}
(5.31) \quad & -\frac{2}{|\nabla u|^2} \sum_i S_{\alpha-1;i} X^{i\bar{i}} \Re \left\{ u_i \eta_{\bar{i}} \right\} \\
& \leq \frac{4(n-\alpha+1)}{|\nabla u|^2} S_{\alpha-1} + \frac{1}{4} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2.
\end{aligned}$$

Substituting (5.31) into (5.26),

$$\begin{aligned}
(5.32) \quad & \theta + \theta \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
& \leq C_5 \left(\frac{1}{A} + \frac{1}{A|\nabla u|} + \frac{1}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
& \quad + \frac{C_6}{|\nabla u|^2} + \frac{C_4}{A|\nabla u|}.
\end{aligned}$$

This leads to a bound for $|\nabla u|$ if A is chosen large enough.

Case 2. $w \leq N$. We have

$$(5.33) \quad \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 |\eta_i|^2 \geq |\nabla \eta|^2 \min_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \geq \frac{C_{n-1}^{\alpha-1}}{w^{\alpha+1}} |\nabla \eta|^2 \geq \frac{C_{n-1}^{\alpha-1}}{N^{\alpha+1}} |\nabla \eta|^2.$$

Substituting (5.25) and (5.33) into (5.24),

$$\begin{aligned}
(5.34) \quad & \left(\epsilon - \frac{C_5}{A} - \frac{C_5}{A|\nabla u|} - \frac{C_5}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \\
& \leq \ln \left(\psi \sum_{\alpha=1}^n c_\alpha S_\alpha \right) + 5n + \frac{C_4}{A|\nabla u|} - \epsilon_1 |\nabla \eta|^2.
\end{aligned}$$

Choosing A sufficiently large and noting the assumption that $|\nabla u| \gg 1$,

$$(5.35) \quad \epsilon_1 |\nabla \eta|^2 + \left(\frac{\epsilon}{2} - \frac{C_5}{|\nabla u|^2} \right) \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \leq C_7.$$

When $|\nabla u|$ is larger than $\sqrt{2C_5/\epsilon}$, we derive a bound for $|\nabla \eta|$. But

$$(5.36) \quad |\nabla u| = |\nabla \mu| \leq |\nabla \eta| + |\nabla v|.$$

□

Applying appropriate $H(t)$, we can obtain time-independent gradient estimate. For Hermitian manifolds, we have the following corollary.

Corollary 5.2. *Under the assumption of Theorem 1.3, there exists a uniform constant C such that*

$$(5.37) \quad |\nabla u|^2 \leq C.$$

Proof. Setting $\mu = \tilde{u}$,

$$(5.38) \quad H(t) = -\frac{\int_M u \omega^n}{\int_M \omega^n}$$

and thus

$$(5.39) \quad \frac{dH}{dt} = -\frac{\int_M \partial_t u \omega^n}{\int_M \omega^n} \geq 0.$$

By theorem 5.1, there are constants C and A such that

$$(5.40) \quad |\nabla u|^2 \leq C e^{A E_2}, \quad E_2 = e^{\sup_{M \times [0, t]} (\tilde{u} - v) - \inf_{M \times [0, t]} (\tilde{u} - v)} - e^{\sup_{M \times [0, t]} (\tilde{u} - v) - (\tilde{u} - v)}.$$

Theorem 4.1 implies that \tilde{u} is uniformly bounded, and then we obtain the bound for $|\nabla u|^2$. □

For Kähler manifolds, we have the following corollary.

Corollary 5.3. *Under the assumption of Theorem 1.5, there exists a uniform constant C such that*

$$(5.41) \quad |\nabla u|^2 \leq C.$$

Proof. Setting $\mu = \hat{u}$,

$$(5.42) \quad H(t) = -\frac{J(u)}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha}$$

and thus

$$(5.43) \quad \frac{dH}{dt} = -\frac{\int_M \partial_t u \left(\sum_{\alpha=1}^n b_\alpha \chi^{n-\alpha} \wedge \omega^\alpha \right)}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha} \geq 0.$$

By theorem 5.1, there are constants C and A such that

$$(5.44) \quad |\nabla u|^2 \leq C e^{A E_2}, \quad E_2 = e^{\sup_{M \times [0, t]} (\hat{u} - v) - \inf_{M \times [0, t]} (\hat{u} - v)} - e^{\sup_{M \times [0, t]} (\hat{u} - v) - (\hat{u} - v)}.$$

By (4.24) in the proof of Lemma 4.4, we have

$$(5.45) \quad \inf_M \hat{u}(x, t) \leq 0 \leq \sup_M \hat{u}(x, t).$$

Then Theorem 4.3 implies that \tilde{u} is uniformly bounded, and consequently we obtain the bound for $|\nabla u|^2$.

□

6. CONVERGENCE OF THE PARABOLIC FLOW

In this section, we prove the convergence of the normalized solution. The arguments of Gill [16] can be applied verbatim here, so we just give a sketch.

Let φ be a positive function on $M \times (0, \infty)$ such that

$$(6.1) \quad \frac{\partial \varphi}{\partial t} = \sum_{i,j} G^{i\bar{j}} \bar{\partial}_j \partial_i \varphi,$$

where

$$(6.2) \quad G^{i\bar{j}} = \frac{\partial}{\partial u_{i\bar{j}}} \left(\ln \frac{S_n(\chi_u)}{\sum_{\alpha=1}^n c_\alpha S_{n-\alpha}(\chi_u)} \right).$$

At a point p and under a normal coordinate chart, $\{G^{i\bar{j}}\}$ is diagonal and

$$(6.3) \quad G^{i\bar{i}} = \sum_{\alpha=1}^n \frac{c_\alpha}{\sum_{\alpha=1}^n c_\alpha S_\alpha} \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2.$$

We have the following Harnack inequality.

Lemma 6.1. *For all $0 < t_1 < t_2$,*

$$(6.4) \quad \sup_{x \in M} \varphi(x, t_1) \leq \inf_{x \in M} \varphi(x, t_2) \left(\frac{t_2}{t_1} \right)^{C_2} e^{\frac{C_3}{t_2 - t_1} + C_1(t_2 - t_1)}$$

Now we apply the argument of Cao [5]. Define $\varphi = \partial_t u$. Let m be a positive integer and define

$$(6.5) \quad \begin{aligned} \varphi'_m(x, t) &= \sup_{y \in M} \varphi(y, m-1) - \varphi(x, m-1+t), \\ \varphi''_m(x, t) &= \varphi(x, m-1+t) - \inf_{y \in M} \varphi(y, m-1). \end{aligned}$$

By (2.21), we have

$$(6.6) \quad \begin{aligned} \frac{\partial \varphi'_m}{\partial t} &= \sum_{i,j} G^{i\bar{j}}(m-1+t) \bar{\partial}_j \partial_i \varphi'_m, \\ \frac{\partial \varphi''_m}{\partial t} &= \sum_{i,j} G^{i\bar{j}}(m-1+t) \bar{\partial}_j \partial_i \varphi''_m. \end{aligned}$$

If $\varphi(x, m-1)$ is not a constant function, φ' and φ'' must be positive functions on $M \times (0, \infty)$ by the maximum principle. Applying Lemma 6.1 with $t_1 = \frac{1}{2}$ and $t_2 = 1$, we have

$$(6.7) \quad O(m-1) + O\left(m - \frac{1}{2}\right) \leq C(O(m-1) - O(m)),$$

where $O(t) = \sup_M \varphi(x, t) - \inf_M \varphi(x, t)$. Therefore $O(t) \leq Ce^{-c_0 t}$ for $c_0 > 0$. On the other hand, this inequality is also true if $\varphi(x, m-1)$ is constant.

For $(x, t) \in M \times [0, \infty)$, there is a point $y \in M$ such that $\partial_t \tilde{u} = 0$ since $\int_M \partial_t \tilde{u} \omega^n = 0$. Therefore,

$$(6.8) \quad |\partial_t \tilde{u}(x, t)| \leq Ce^{-c_0 t},$$

and consequently

$$(6.9) \quad \partial_t Q \leq 0,$$

where $Q = \tilde{u} + \frac{C}{c_0} e^{-c_0 t}$. Since Q is bounded and decreasing,

$$(6.10) \quad \lim_{t \rightarrow \infty} \tilde{u} = \lim_{t \rightarrow \infty} Q = \tilde{u}_\infty.$$

By contradiction argument, we can prove that the convergence of \tilde{u} to \tilde{u}_∞ is actually C^∞ .

Note that \tilde{u} satisfies

$$(6.11) \quad \frac{\partial \tilde{u}}{\partial t} + \frac{\int_M \partial_t u \omega^n}{\int_M \omega^n} = \ln \frac{\chi_{\tilde{u}}^n}{\sum_{\alpha=1}^n b_\alpha \chi_{\tilde{u}}^{n-\alpha} \wedge \omega^\alpha} - \ln \psi.$$

Letting $t \rightarrow \infty$, the right side of (6.11) tells us that it converges. Then (6.8) implies that it converges to a constant b . This completes the proof of the convergence.

7. GENERALIZED J -FLOW IN KÄHLER GEOMETRY

In this section, we just show some key steps of the proof, that is, the second order estimate and the uniform estimate. The remaining parts follow from the previous sections.

Rewriting (1.19),

$$(7.1) \quad \frac{\partial u}{\partial t} = \frac{1}{c} - \sum_{\alpha=1}^n c_\alpha S_\alpha.$$

Since ω is Kähler,

$$(7.2) \quad X_{\bar{i}j\bar{j}} - X_{j\bar{j}\bar{i}} = R_{j\bar{j}\bar{i}} X_{\bar{i}} - R_{\bar{i}j\bar{j}} X_{j\bar{j}} - G_{\bar{i}j\bar{j}},$$

where

$$(7.3) \quad G_{\bar{i}\bar{j}\bar{j}} = \chi_{j\bar{j}\bar{i}} - \chi_{\bar{i}\bar{j}\bar{j}} + \sum_p R_{j\bar{j}i\bar{p}} \chi_{p\bar{i}} - \sum_p R_{\bar{i}\bar{j}j\bar{p}} \chi_{p\bar{j}}.$$

Differentiating the equation at p

$$(7.4) \quad \partial_t(\partial_t u) = \sum_i S_{\alpha-1;i}(X^{\bar{i}\bar{i}})^2(\partial_t u)_{\bar{i}\bar{i}},$$

$$(7.5) \quad \partial_t u_l = \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i}(X^{\bar{i}\bar{i}})^2 X_{\bar{i}\bar{l}},$$

and

$$(7.6) \quad \partial_t u_{\bar{l}\bar{l}} \leq \sum_{\alpha=1}^n c_\alpha \left[\sum_i S_{\alpha-1;i}(X^{\bar{i}\bar{i}})^2 X_{\bar{i}\bar{l}\bar{l}} - \sum_{i,j} S_{\alpha-1;i}(X^{\bar{i}\bar{i}})^2 X^{j\bar{j}} X_{j\bar{i}\bar{l}} X_{i\bar{j}l} \right].$$

7.1. Second order estimate.

Theorem 7.1. *Let $u \in C^4(M \times [0, T])$ be an admissible solution to equation (1.19) and $w = \Delta u + \text{tr} \chi$. Then there are uniform constants C and A such that*

$$(7.7) \quad w \leq C e^{A(u - \inf_{M \times [0, t]} u)},$$

where C, A depend only on geometric data.

Proof. We consider the function $w e^\phi$ where ϕ , where

$$(7.8) \quad \phi := -A(u - v).$$

and A is to be determined. Suppose that $\ln w e^\phi$ achieves its maximum at some point $(p, t_0) \in M_t = M \times (0, t]$. Without loss of generality, we may assume that $w \gg 1$. Choose a local chart around p such that $g_{i\bar{j}} = \delta_{ij}$ and $X_{i\bar{j}}$ is diagonal at p when $t = t_0$. Therefore, we have at the point (p, t_0) ,

$$(7.9) \quad \frac{\partial_l w}{w} + \partial_l \phi = 0,$$

$$(7.10) \quad \frac{\bar{\partial}_l w}{w} + \bar{\partial}_l \phi = 0,$$

$$(7.11) \quad \frac{\partial_t w}{w} + \partial_t \phi \geq 0,$$

and

$$(7.12) \quad \frac{\bar{\partial}_l \partial_l w}{w} - \frac{|\partial_l w|^2}{w^2} + \bar{\partial}_l \partial_l \phi \leq 0.$$

Note that

$$(7.13) \quad \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \frac{|\partial_i w|^2}{w} \leq \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{i\bar{j}l} X_{j\bar{i}l}.$$

By (7.2), (7.13) and (3.19),

$$(7.14) \quad \begin{aligned} & \sum_{i,j,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X^{j\bar{j}} X_{j\bar{i}l} X_{i\bar{j}l} - \sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 X_{i\bar{i}l} \\ & \geq w \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi - C_1 w \sum_{i,l} S_{\alpha-1;i} (X^{i\bar{i}})^2 - C_2 \frac{\alpha}{n} S_\alpha. \end{aligned}$$

Combining (7.6) and (7.14),

$$(7.15) \quad \begin{aligned} \partial_t w & \leq -w \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi \\ & \quad + C_1 w \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_2 \sup_M \sum_{\alpha=1}^n c_\alpha S_\alpha (\chi^{-1}). \end{aligned}$$

By (7.10) and (7.11),

$$(7.16) \quad w \partial_t \phi \geq w \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \bar{\partial}_i \partial_i \phi - C_1 w \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 - C_3.$$

Consequently,

$$(7.17) \quad \begin{aligned} & Aw \left(\sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (v_{i\bar{i}} - u_{i\bar{i}}) + \frac{1}{c} - \sum_{\alpha=1}^n c_\alpha S_\alpha \right) \\ & \leq C_1 w \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_3. \end{aligned}$$

Similar to Lemma 3.1, we can show that there are constants $N, \theta > 0$ such that when $w \geq N$,

$$(7.18) \quad \begin{aligned} & \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 (v_{i\bar{i}} - u_{i\bar{i}}) - \frac{1}{c} + \sum_{\alpha=1}^n c_\alpha S_\alpha \\ & \geq \theta + \theta \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2. \end{aligned}$$

If $w \geq N$, from (7.17)

$$\begin{aligned}
 (7.19) \quad & Aw \left(\theta + \theta \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 \right) \\
 & \leq C_1 w \sum_{\alpha=1}^n c_\alpha \sum_i S_{\alpha-1;i} (X^{i\bar{i}})^2 + C_3.
 \end{aligned}$$

Choosing $A > \frac{C_1}{\theta}$, we derive a bound for w .

□

7.2. Uniform estimate.

Theorem 7.2. *Under the assumption of Theorem 1.6, there exists a uniform constant C such that*

$$(7.20) \quad \sup_M u(x, t) - \inf_M u(x, t) \leq C.$$

Proof. Different from Section 4, we shall use the well known functional I defined by

$$(7.21) \quad I = J_0 = \int_0^1 \int_M \frac{\partial v}{\partial s} \chi_v^n ds,$$

for any path $v(s) \subset \mathcal{H}$ connecting 0 and u . Then

$$(7.22) \quad I(u(T)) = \int_0^T \int_M \partial_t u \chi_u^n dt = 0.$$

It is shown by Weinkove [40] that equality (7.22) implies

$$(7.23) \quad 0 \leq \sup_M u(x, t) \leq C_1 - C_2 \inf_M u(x, t).$$

Therefore, we only need to prove the lower bound of $u(x, t)$. If such a lower bound does not exist, then we can choose a sequence $t_i \rightarrow \infty$ such that

$$(7.24) \quad \inf_M u(x, t_i) = \inf_{t \in [0, t_i]} \inf_M u(x, t),$$

and

$$(7.25) \quad \inf_M u(x, t_i) \rightarrow -\infty.$$

By theorem 7.1,

$$\begin{aligned}
 (7.26) \quad & w(x, t_i) \leq C e^{A(u(x, t_i) - \inf_{M \times [0, t_i]} u)} \\
 & = C e^{A(u(x, t_i) - \inf_M u(x, t_i))}.
 \end{aligned}$$

As shown in [34], it follows that

$$(7.27) \quad \sup_M u(x, t_i) - \inf_M u(x, t_i) \leq C$$

for some positive constant C , which contradicts our assumption. \square

7.3. Lower bound and properness. We consider

$$(7.28) \quad \hat{J}_\chi(u) := \sum_{\alpha=1}^n b_\alpha J_\alpha(u) - cI(u), \quad c = \frac{\int_M \chi^n}{\sum_{\alpha=1}^n b_\alpha \int_M \chi^{n-\alpha} \wedge \omega^\alpha}.$$

The generalized J -flow is the gradient flow for \hat{J}_χ .

Choosing a path $\phi(t)$, we have

$$(7.29) \quad \frac{d\hat{J}_\chi(\phi)}{dt} = \int_M \dot{\phi} \left(\sum_{\alpha=1}^n b_\alpha \chi_\phi^{n-\alpha} \wedge \omega^\alpha - c \chi_\phi^n \right)$$

and

$$(7.30) \quad \begin{aligned} \frac{d^2 \hat{J}_\chi(\phi)}{dt^2} &= \int_M \ddot{\phi} \left(\sum_{\alpha=1}^n b_\alpha \chi_\phi^{n-\alpha} \wedge \omega^\alpha - c \chi_\phi^n \right) \\ &\quad - \frac{1}{2} \int_M \sqrt{-1} \partial \dot{\phi} \wedge \bar{\partial} \dot{\phi} \wedge \left(\sum_{\alpha=1}^n b_\alpha (n-\alpha) \chi_\phi^{n-\alpha-1} \wedge \omega^\alpha - c n \chi_\phi^{n-1} \right). \end{aligned}$$

Similar to the work of Mabuchi [24], the class of functional \hat{J}_χ has 1-cocycle condition in \mathcal{H} by the independence of path. Together with (7.29) and (7.30), we have the following theorem.

Theorem 7.3. *There is at most one Kähler metric $\chi' \in [\chi]$ such that*

$$(7.31) \quad \sum_{\alpha=1}^n b_\alpha \chi'^{n-\alpha} \wedge \omega^\alpha = c \chi'^n.$$

In addition, if (7.31) holds true for $\chi' = \chi_u$, then \hat{J}_χ has a minimum value at u .

Proof. The 1-cocycle condition tells us that $\hat{J}_\chi(v) = \hat{J}_\chi(u) + \hat{J}_{\chi_u}(v)$. If there are two metrics $\chi' = \chi_u$ and $\chi'' = \chi_{u'}$ satisfying equation (7.31). Applying (7.29) and (7.29) to \hat{J}_{χ_u} with $\phi(t) = t(u' - u)$,

$$(7.32) \quad \left. \frac{d\hat{J}_{\chi'}(\phi)}{dt} \right|_{t=0} = \left. \frac{d\hat{J}_{\chi'}(\phi)}{dt} \right|_{t=1} = 0,$$

and

$$(7.33) \quad \begin{aligned} \frac{d^2 \hat{J}_\chi(\phi)}{dt^2} &= -\frac{1}{2} \int_M \sqrt{-1} \partial \dot{\phi} \wedge \bar{\partial} \dot{\phi} \wedge \left(\sum_{\alpha=1}^n b_\alpha (n-\alpha) \chi_\phi^{n-\alpha-1} \wedge \omega^\alpha - cn \chi_\phi^{n-1} \right) \\ &\geq \delta \int_M \sqrt{-1} \partial(u-u') \wedge \bar{\partial}(u-u') \wedge \chi_\phi^{n-1}. \end{aligned}$$

So it must be that $u - u'$ is constant.

For any path ϕ , we have

$$(7.34) \quad \left. \frac{d \hat{J}_{\chi'}(\phi)}{dt} \right|_{t=0} = 0,$$

and

$$(7.35) \quad \left. \frac{d^2 \hat{J}_{\chi'}(\phi)}{dt^2} \right|_{t=0} > 0.$$

$$\text{So } \hat{J}_\chi(v) = \hat{J}_\chi(u) + \hat{J}_{\chi'}(v) \geq \hat{J}_\chi(u)$$

□

An alternative approach is probably the method by Chen [6, 7] using the $C^{1,1}$ geodesics in \mathcal{H} .

We recall the definition of properness in [10].

Definition 7.4. We say that \hat{J}_χ is proper, if there are constants $C, \delta > 0$ such that for any $\chi_u > 0$,

$$(7.36) \quad \hat{J}_\chi(u) \geq -C + \delta \int_M u(\chi^n - \chi_u^n).$$

A corollary follows from Theorem 1.5 (or Theorem 1.6), Theorem 7.3 and the perturbation method in [10].

Theorem 7.5. *Suppose that the cone condition (1.11) holds true for $\psi = c$. Then the functional \hat{J}_χ is bounded from below and then proper.*

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